特征值和特征向量

Eigenvalues and Eigenvectors

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Elementary Properties of Eigensystems

- Up to this point, almost everything was either motivated by or evolved from the consideration of systems of linear algebraic equations.
- Many topics will be motivated or driven by applications involving systems of linear differential equations and their discrete counterparts, difference equations.
- For example, consider the problem of solving the system of two first-order linear differential equations, $du_1/dt = 7u_1 - 4u_2$ and $du_2/dt = 5u_1 - 2u_2$.
- In matrix notation, this system is

$$\begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 7 & -4 \\ 5 & -2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \quad or, \quad \mathbf{u}' = \mathbf{A}\mathbf{u}.$$

- Because solutions of a single equation $mu' = \lambda u$ have the form $u = \alpha e^{\lambda t}$, we are motivated to seek solutions with form

$$u_1 = \alpha_1 e^{\lambda t} \quad and \quad u_2 = \alpha_2 e^{\lambda t}.$$

- Differentiating these two expression and substituting the result yields $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$, where $\mathbf{x} = (\alpha_1, \alpha_2)^T$.

$$\begin{aligned} \alpha_1 \lambda e^{\lambda t} &= 7\alpha_1 e^{\lambda t} - 4\alpha_2 e^{\lambda t} \\ \alpha_2 \lambda e^{\lambda t} &= 5\alpha_1 e^{\lambda t} - 2\alpha_2 e^{\lambda t} \end{aligned} \Rightarrow \begin{aligned} \alpha_1 \lambda &= 7\alpha_1 - 4\alpha_2 \\ \alpha_2 \lambda &= 5\alpha_1 - 2\alpha_2 \end{aligned} \Rightarrow \begin{pmatrix} 7 & -4 \\ 5 & -2 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \lambda \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}.$$

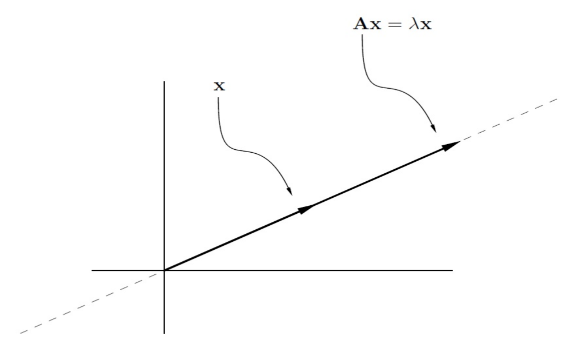
- What we really need are scalars λ and nonzero vectors $\mathbf{x}$ that satisfy $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$.

Eigenvalues and Eigenvectors

For an $n \times n$ matrix $\mathbf{A}$, scalars λ and vectors $\mathbf{x}_{n\times 1} \neq \mathbf{0}$ satisfying $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$ are called ***eigenvalues*** and ***eigenvectors*** of $\mathbf{A}$, respectively, and any such pair, $(\lambda, \mathbf{x})$, is called an ***eigenpair*** for $\mathbf{A}$. The set of *distinct* eigenvalues, denoted by $\sigma(\mathbf{A})$, is called the ***spectrum*** of $\mathbf{A}$.

- $\lambda \in \sigma(\mathbf{A}) \iff \mathbf{A} - \lambda\mathbf{I}$ is singular $\iff \det(\mathbf{A} - \lambda\mathbf{I}) = 0$.
- $\{\mathbf{x} \neq \mathbf{0} \mid \mathbf{x} \in N(\mathbf{A} - \lambda\mathbf{I})\}$ is the set of all eigenvectors associated with λ. From now on, $N(\mathbf{A} - \lambda\mathbf{I})$ is called an ***eigenspace*** for $\mathbf{A}$.
- Nonzero row vectors $\mathbf{y}^*$ such that $\mathbf{y}^*(\mathbf{A} - \lambda\mathbf{I}) = \mathbf{0}$ are called ***left-hand eigenvectors*** for $\mathbf{A}$.

- Geometrically, $\mathbf{Ax} = \lambda \mathbf{x}$ says that under transformation by $\mathbf{A}$, eigenvectors experience only changes in magnitude or sign-the orientation of $\mathbf{Ax}$ in $\Re^n$ is the same as that of $\mathbf{x}$.
- The eigenvalue λ is simply the amount of "stretch" or "shrink" to which the eigenvector $\mathbf{x}$ is subjected when transformed by $\mathbf{A}$.



Characteristic Polynomial and Equation

- The ***characteristic polynomial*** of $\mathbf{A}_{n\times n}$ is $p(\lambda) = \det(\mathbf{A} - \lambda\mathbf{I})$. The degree of $p(\lambda)$ is n, and the leading term in $p(\lambda)$ is $(-1)^n\lambda^n$.
- The ***characteristic equation*** for $\mathbf{A}$ is $p(\lambda) = 0$.
- The eigenvalues of $\mathbf{A}$ are the solutions of the characteristic equation or, equivalently, the roots of the characteristic polynomial.
- Altogether, $\mathbf{A}$ has n eigenvalues, but some may be complex numbers (even if the entries of $\mathbf{A}$ are real numbers), and some eigenvalues may be repeated.
- If $\mathbf{A}$ contains only real numbers, then its complex eigenvalues must occur in conjugate pairs—i.e., if $\lambda \in \sigma(\mathbf{A})$, then $\overline{\lambda} \in \sigma(\mathbf{A})$.

Problem: Determine the eigenvalues and eigenvectors of $\mathbf{A} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$.

Solution: The characteristic polynomial is

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} 1-\lambda & -1 \\ 1 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 + 1 = \lambda^2 - 2\lambda + 2,$$

so the characteristic equation is $\lambda^2 - 2\lambda + 2 = 0$. Application of the quadratic formula yields

$$\lambda = \frac{2 \pm \sqrt{-4}}{2} = \frac{2 \pm 2\sqrt{-1}}{2} = 1 \pm \mathrm{i},$$

so the spectrum of $\mathbf{A}$ is $\sigma(\mathbf{A}) = \{1+\mathrm{i},\, 1-\mathrm{i}\}$. Notice that the eigenvalues are complex conjugates of each other—as they must be because complex eigenvalues of real matrices must occur in conjugate pairs. Now find the eigenspaces.

For $\lambda = 1 + \mathrm{i}$,

$$\mathbf{A} - \lambda \mathbf{I} = \begin{pmatrix} -\mathrm{i} & -1 \\ 1 & -\mathrm{i} \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -\mathrm{i} \\ 0 & 0 \end{pmatrix} \implies N(\mathbf{A} - \lambda \mathbf{I}) = span\left\{ \begin{pmatrix} \mathrm{i} \\ 1 \end{pmatrix} \right\}.$$

For $\lambda = 1 - \mathrm{i}$,

$$\mathbf{A} - \lambda \mathbf{I} = \begin{pmatrix} \mathrm{i} & -1 \\ 1 & \mathrm{i} \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & \mathrm{i} \\ 0 & 0 \end{pmatrix} \implies N(\mathbf{A} - \lambda \mathbf{I}) = span\left\{ \begin{pmatrix} -\mathrm{i} \\ 1 \end{pmatrix} \right\}.$$

- As we have seen, computing eigenvalues boils down to solving a polynomial equation. But determining solutions to polynomial equations can be a formidable task.
- It was proven that it's impossible to express the roots of a general polynomial of degree five or higher using radicals of the coefficients.
- This means that there does not exist a generalized version of the quadratic formula for polynomials of degree greater than four.
- And general polynomial equations can not be solved by a finite number of arithmetic operations involving $+, -, \times, \div, \sqrt[n]{\ }$.
- The eigenvalue problem generally requires an infinite algorithm, so all practical eigenvalue computations are accomplished by iterative methods.

- Recall that an $r \times r$ principal submatrix of $\mathbf{A}_{n\times n}$ is a submatrix that lies on the same set of r rows and columns, and an $r \times r$ principal minor if the determinant of an $r \times r$ principal submatrix.
- Related to the principal minors are the symmetric functions of the eigenvalues.
- The k^{th} symmetric function of $\lambda_1, \lambda_2, \cdots, \lambda_n$ is defined to be sum of the product of the eigenvalues taken k at a time. That is

$$s_k = \sum_{1\le i_1<\cdots<i_k\le n} \lambda_{i_1} \cdots \lambda_{i_k}.$$

- For example, when $n = 4$,

$$\begin{aligned}
s_1 &= \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4, \\
s_2 &= \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_1\lambda_4 + \lambda_2\lambda_3 + \lambda_2\lambda_4 + \lambda_3\lambda_4, \\
s_3 &= \lambda_1\lambda_2\lambda_3 + \lambda_1\lambda_2\lambda_4 + \lambda_1\lambda_3\lambda_4 + \lambda_2\lambda_3\lambda_4, \\
s_4 &= \lambda_1\lambda_2\lambda_3\lambda_4.
\end{aligned}$$

Coefficients in the Characteristic Equation

If $\lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + \cdots + c_{n-1}\lambda + c_n = 0$ is the characteristic equation for $\mathbf{A}_{n\times n}$, and if s_k is the k^{th} symmetric function of the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of $\mathbf{A}$, then

- $c_k = (-1)^k \sum(\text{all } k\times k \text{ principal minors})$,
- $s_k = \sum(\text{all } k\times k \text{ principal minors})$,
- $trace\,(\mathbf{A}) = \lambda_1 + \lambda_2 + \cdots + \lambda_n = -c_1$,
- $\det(\mathbf{A}) = \lambda_1\lambda_2\cdots\lambda_n = (-1)^n c_n$.

- **Problem:** Determine the eigenvalues and eigenvectors of

$$\mathbf{A} = \begin{pmatrix} -3 & 1 & -3 \\ 20 & 3 & 10 \\ 2 & -2 & 4 \end{pmatrix}.$$

- **Solution:** Use the principal minors to obtain the characteristic equation

$$\lambda^3 - 4\lambda^2 - 3\lambda + 18 = 0.$$

- A result from elementary algebra states that if the coefficients α_i in

$$\lambda^n + \alpha_{n-1}\lambda^{n-1} + \cdots + \alpha_1\lambda + \alpha_0$$

are integers, then every integer solution is a factor of α_0.
  - This means that integer eigenvalues must be contained in the set

$$\mathcal{S} = \{\pm 1, \pm 2, \pm 3, \pm 6, \pm 9, \pm 18\}.$$

  - Evaluating $p(\lambda) = \det(\mathbf{A} - \lambda\mathbf{I})$ for each $\lambda \in \mathcal{S}$ reveals that $p(3) = 0$ and $p(-2) = 0$, so $\lambda = 3$ and $\lambda = -2$ are eigenvalues for $\mathbf{A}$.
  - So the spectrum of $\mathbf{A}$ is $\sigma(\mathbf{A}) = \{3, -2\}$ in which the algebraic multiplicity of $\lambda = 3$ is two.

For $\lambda = 3$,

$$\mathbf{A} - 3\mathbf{I} \longrightarrow \begin{pmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \implies N(\mathbf{A} - 3\mathbf{I}) = span\left\{\begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}\right\}.$$

For $\lambda = -2$,

$$\mathbf{A} + 2\mathbf{I} \longrightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix} \implies N(\mathbf{A} + 2\mathbf{I}) = span\left\{\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}\right\}.$$

- **Spectral Radius** For square matrices $\mathbf{A}$, the number

$$\rho(\mathbf{A}) = \max_{\lambda \in \sigma(\mathbf{A})} |\lambda|$$

  is called the spectral radius of $\mathbf{A}$.
- it is easy observing that $\rho(\mathbf{A}) \leq \|\mathbf{A}\|$ for every matrix norm.

Gerschgorin Circles

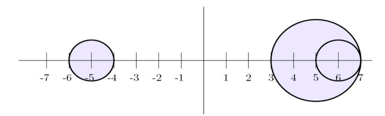
- The eigenvalues of $\mathbf{A} \in \mathcal{C}^{n\times n}$ are contained the union $\mathcal{G}_r$ of the n ***Gerschgorin circles*** defined by

$$|z - a_{ii}| \leq r_i, \quad \text{where} \quad r_i = \sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ij}| \text{ for } i = 1, 2, \ldots, n.$$

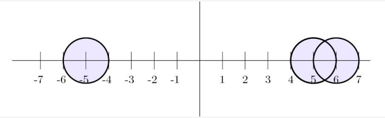
  In other words, the eigenvalues are trapped in the collection of circles centered at a_{ii} with radii given by the sum of absolute values in $\mathbf{A}_{i*}$ with a_{ii} deleted.
- Furthermore, if a union $\mathcal{U}$ of k Gerschgorin circles does not touch any of the other $n-k$ circles, then there are exactly k eigenvalues (counting multiplicities) in the circles in $\mathcal{U}$.
- Since $\sigma(\mathbf{A}^T) = \sigma(\mathbf{A})$, the deleted absolute row sums can be replaced by deleted absolute column sums, so the eigenvalues of $\mathbf{A}$ are also contained in the union $\mathcal{G}_c$ of the circles defined by

$$|z - a_{jj}| \leq c_j, \quad \text{where} \quad c_j = \sum_{\substack{i=1 \\ i\neq j}}^{n} |a_{ij}| \text{ for } j = 1, 2, \ldots, n.$$

- Combining the above two equations means the eigenvalues of $\mathbf{A}$ are contained in the intersection $\mathcal{G}_r \bigcap \mathcal{G}_c$.
- **Problem:** Estimate the eigenvalues of $\mathbf{A} = \begin{pmatrix} 5 & 1 & 1 \\ 0 & 6 & 1 \\ 1 & 0 & -5 \end{pmatrix}$.
  - A crude estimate is derived from the bound $|\lambda| \leq \|\mathbf{A}\|_\infty = 7$.
  - Better estimates are produced by the Gerschgorin circle, which guarantee that one eigenvalue is in (or on) the circle centered at -5, which the remaining two eigenvalues are in (or on) the larger circle centered at $+5$.



  - by considering $\mathcal{G}_r \bigcap \mathcal{G}_c$, one eigenvalue is in the circle centered at -5, while the other two eigenvalues are in the union of the other two circle



Diagonalization by Similarity Transformations

- The correct choice of a coordinate system (or basis) often can simplify the form of an equation or the analysis of a particular problem or the analysis of a particular problem.
- For a linear operator $\mathbf{L}$ on a finite-dimensional space $\mathcal{V}$, the goal is to find a basis $\mathcal{B}$ such that the matrix representation $\mathbf{L}$ is as simple as possible.
- Since different matrix representation $\mathbf{A}$ and $\mathbf{B}$ of $\mathbf{L}$ are related by a similarity transformation $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{B}$, the fundamental problem is to find a nonsingular matrix $\mathbf{P}$ such that $\mathbf{P}^{-1}\mathbf{A}\mathbf{P}$ is as simple as possible.

Similarity

- Two $n \times n$ matrices $\mathbf{A}$ and $\mathbf{B}$ are said to be ***similar*** whenever there exists a nonsingular matrix $\mathbf{P}$ such that $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{B}$. The product $\mathbf{P}^{-1}\mathbf{A}\mathbf{P}$ is called a ***similarity transformation*** on $\mathbf{A}$.
- **A Fundamental Problem.** Given a square matrix $\mathbf{A}$, reduce it to the simplest possible form by means of a similarity transformation.

- Diagonal matrices have the simplest form, so we first ask, "Is every square matrix similar to a diagonal matrix?"
- Linear algebra and matrix theory would be simpler subjects if this were true, but it's not.
- Nilpotent matrices are not similar to a diagonal matrix, which are not the only ones that can't diagonalized.
- So, if not all square matrices can be diagonalized by a similarity transformation, what are the characteristics of those that can?

Diagonalizability

- A square matrix $\mathbf{A}$ is said to be ***diagonalizable*** whenever $\mathbf{A}$ is similar to a diagonal matrix.
- A ***complete set of eigenvectors*** for $\mathbf{A}_{n\times n}$ is any set of n linearly independent eigenvectors for $\mathbf{A}$. Not all matrices have complete sets of eigenvectors.
  Matrices that fail to possess complete sets of eigenvectors are sometimes called ***deficient*** or ***defective*** matrices.
- $\mathbf{A}_{n\times n}$ is diagonalizable if and only if $\mathbf{A}$ possesses a complete set of eigenvectors. Moreover, $\mathbf{P}^{-1}\mathbf{AP} = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ if and only if the columns of $\mathbf{P}$ constitute a complete set of eigenvectors and the λ_j 's are the associated eigenvalues—i.e., each $(\lambda_j, \mathbf{P}_{*j})$ is an eigenpair for $\mathbf{A}$.

Problem: If possible, diagonalize the following matrix with a similarity transformation:

$$\mathbf{A} = \begin{pmatrix} 1 & -4 & -4 \\ 8 & -11 & -8 \\ -8 & 8 & 5 \end{pmatrix}.$$

Solution: Determine whether or not $\mathbf{A}$ has a complete set of three linearly independent eigenvectors. The characteristic equation—perhaps computed by

$$\lambda^3 + 5\lambda^2 + 3\lambda - 9 = (\lambda - 1)(\lambda + 3)^2 = 0.$$

Therefore, $\lambda = 1$ is a simple eigenvalue, and $\lambda = -3$ is repeated twice (we say its algebraic multiplicity is 2). Bases for the eigenspaces $N(\mathbf{A} - 1\mathbf{I})$ and $N(\mathbf{A} + 3\mathbf{I})$ are determined in the usual way to be

$$N(\mathbf{A} - 1\mathbf{I}) = span\left\{\begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}\right\} \quad \text{and} \quad N(\mathbf{A} + 3\mathbf{I}) = span\left\{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}\right\},$$

and it's easy to check that when combined these three eigenvectors constitute a linearly independent set. Consequently, $\mathbf{A}$ must be diagonalizable. To explicitly exhibit the similarity transformation that diagonalizes $\mathbf{A}$, set

$$\mathbf{P} = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}, \quad \text{and verify} \quad \mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{pmatrix} = \mathbf{D}.$$

Similarity Preserves Eigenvalues

Row reductions don't preserve eigenvalues (try a simple example). However, similar matrices have the same characteristic polynomial, so they have the same eigenvalues with the same multiplicities. **Caution!** Similar matrices need not have the same eigenvectors

- In the context of linear operators, this means that eigenvalues of a matrix representation of an operator $\mathbf{L}$ are in variant under a change of basis.
- In other words, the eigenvalues are intrinsic to $\mathbf{L}$ in the sense that they are independent of any coordinate representation.
- Since not all square matrices are diagonlizable, can every square matrix be triangularized similarity?
- The answer is yes! In fact, as Issai Schur realized in 1909, the similarity transformation always can be made to be unitary.

Schur's Triangularization Theorem

Every square matrix is unitarily similar to an upper-triangular matrix. That is, for each $\mathbf{A}_{n\times n}$, there exists a unitary matrix $\mathbf{U}$ (not unique) and an upper-triangular matrix $\mathbf{T}$ (not unique) such that $\mathbf{U}^*\mathbf{A}\mathbf{U} = \mathbf{T}$, and the diagonal entries of $\mathbf{T}$ are the eigenvalues of $\mathbf{A}$.

- Schur's theorem is not the complete story on triangularizing by similarity.
- By allowing nonunitary similarity transformations, the structure of the uppertriangular matrix $\mathbf{T}$ can be simplified to contain zeros everywhere except on the diagonal and the superdiagonal.

Multiplicities

For $\lambda \in \sigma(\mathbf{A}) = \{\lambda_1, \lambda_2, \ldots, \lambda_s\}$, we adopt the following definitions.

- The ***algebraic multiplicity*** of λ is the number of times it is repeated as a root of the characteristic polynomial. In other words, $alg\ mult_{\mathbf{A}}(\lambda_i) = a_i$ if and only if $(x-\lambda_1)^{a_1}\cdots(x-\lambda_s)^{a_s} = 0$ is the characteristic equation for $\mathbf{A}$.
- When $alg\ mult_{\mathbf{A}}(\lambda) = 1$, λ is called a ***simple eigenvalue.***
- The ***geometric multiplicity*** of λ is $\dim N(\mathbf{A} - \lambda\mathbf{I})$. In other words, $geo\ mult_{\mathbf{A}}(\lambda)$ is the maximal number of linearly independent eigenvectors associated with λ.
- Eigenvalues such that $alg\ mult_{\mathbf{A}}(\lambda) = geo\ mult_{\mathbf{A}}(\lambda)$ are called ***semisimple eigenvalues*** of $\mathbf{A}$.

  a simple eigenvalue is always semisimple, but not conversely.

Multiplicity Inequality

For every $\mathbf{A} \in \mathcal{C}^{n\times n}$, and for each $\lambda \in \sigma(\mathbf{A})$,

$$geo\ mult_{\mathbf{A}}(\lambda) \leq alg\ mult_{\mathbf{A}}(\lambda).$$

- Determining whether or not $\mathbf{A}$ is diagonalizable is equivalent to determining whether or not $\mathbf{A}$ has a complete linearly independent set of eigenvectors.
- This can be done if you are willing and able to compute all of the eigenvalues and eigenvectors for $\mathbf{A}$.
- But this brute force approach can be a monumental task.
- Fortunately, there are some theoretical tools to help determine how many linearly independent eigenvectors a given matrix possesses.

Independent Eigenvectors

Let $\{\lambda_1, \lambda_2, \ldots, \lambda_k\}$ be a set of distinct eigenvalues for $\mathbf{A}$.

- If $\{(\lambda_1, \mathbf{x}_1), (\lambda_2, \mathbf{x}_2), \ldots, (\lambda_k, \mathbf{x}_k)\}$ is a set of eigenpairs for $\mathbf{A}$, then $\mathcal{S} = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k\}$ is a linearly independent set.
- If $\mathcal{B}_i$ is a basis for $N(\mathbf{A} - \lambda_i \mathbf{I})$, then $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \cdots \cup \mathcal{B}_k$ is a linearly independent set.

- These results lead to the following characterization of diagonalizability.

Diagonalizability and Multiplicities

A matrix $\mathbf{A}_{n\times n}$ is diagonalizable if and only if

$$geo\ mult_{\mathbf{A}}(\lambda) = alg\ mult_{\mathbf{A}}(\lambda)$$

for each $\lambda \in \sigma(\mathbf{A})$ —i.e., if and only if every eigenvalue is semisimple.

Problem: Determine if either of the following matrices is diagonalizable:

$$\mathbf{A} = \begin{pmatrix} -1 & -1 & -2 \\ 8 & -11 & -8 \\ -10 & 11 & 7 \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} 1 & -4 & -4 \\ 8 & -11 & -8 \\ -8 & 8 & 5 \end{pmatrix}.$$

Solution: Each matrix has exactly the same characteristic equation

$$\lambda^3 + 5\lambda^2 + 3\lambda - 9 = (\lambda - 1)(\lambda + 3)^2 = 0,$$

so $\sigma(\mathbf{A}) = \{1, -3\} = \sigma(\mathbf{B})$, where $\lambda = 1$ has algebraic multiplicity 1 and $\lambda = -3$ has algebraic multiplicity 2. Since

$$geo\ mult_{\mathbf{A}}(-3) = \dim N(\mathbf{A} + 3\mathbf{I}) = 1 < alg\ mult_{\mathbf{A}}(-3),$$

$\mathbf{A}$ is *not* diagonalizable. On the other hand,

$$geo\ mult_{\mathbf{B}}(-3) = \dim N(\mathbf{B} + 3\mathbf{I}) = 2 = alg\ mult_{\mathbf{B}}(-3),$$

and $geo\ mult_{\mathbf{B}}(1) = 1 = alg\ mult_{\mathbf{B}}(1)$, so $\mathbf{B}$ *is* diagonalizable.

Distinct Eigenvalues

If no eigenvalue of $\mathbf{A}$ is repeated, then $\mathbf{A}$ is diagonalizable.
Caution! The converse is not true

- An elegant and more geometrical way of expressing diagonalizability is now presented to help simplify subsequent analyses and pave the way for extensions.

Spectral Theorem for Diagonalizable Matrices

A matrix $\mathbf{A}_{n\times n}$ with spectrum $\sigma(\mathbf{A}) = \{\lambda_1, \lambda_2, \ldots, \lambda_k\}$ is diagonalizable if and only if there exist matrices $\{\mathbf{G}_1, \mathbf{G}_2, \ldots, \mathbf{G}_k\}$ such that

$$\mathbf{A} = \lambda_1\mathbf{G}_1 + \lambda_2\mathbf{G}_2 + \cdots + \lambda_k\mathbf{G}_k,$$

where the $\mathbf{G}_i$'s have the following properties.

- $\mathbf{G}_i$ is the projector onto $N(\mathbf{A} - \lambda_i\mathbf{I})$ along $R(\mathbf{A} - \lambda_i\mathbf{I})$.
- $\mathbf{G}_i\mathbf{G}_j = \mathbf{0}$ whenever $i \neq j$.
- $\mathbf{G}_1 + \mathbf{G}_2 + \cdots + \mathbf{G}_k = \mathbf{I}$.

The expansion is known as the ***spectral decomposition*** of $\mathbf{A}$, and the $\mathbf{G}_i$'s are called the ***spectral projectors*** associated with $\mathbf{A}$.

If $\mathbf{A}$ is diagonalizable, and if $\mathbf{X}_i$ is a matrix whose columns form a basis for $N(\mathbf{A}-\lambda_i\mathbf{I})$, then $\mathbf{P}=\left(\mathbf{X}_1\,|\,\mathbf{X}_2\,|\cdots|\,\mathbf{X}_k\right)$ is nonsingular. If $\mathbf{P}^{-1}$ is partitioned in a conformable manner, then we must have

$$\mathbf{A}=\mathbf{P}\mathbf{D}\mathbf{P}^{-1}=\left(\mathbf{X}_1\,|\,\mathbf{X}_2\,|\cdots|\,\mathbf{X}_k\right)\begin{pmatrix}\lambda_1\mathbf{I} & \mathbf{0} & \cdots & \mathbf{0}\\ \mathbf{0} & \lambda_2\mathbf{I} & \cdots & \mathbf{0}\\ \vdots & \vdots & \ddots & \vdots\\ \mathbf{0} & \mathbf{0} & \cdots & \lambda_k\mathbf{I}\end{pmatrix}\begin{pmatrix}\mathbf{Y}_1^T\\ \hline \mathbf{Y}_2^T\\ \hline \vdots\\ \hline \mathbf{Y}_k^T\end{pmatrix}$$

$$=\lambda_1\mathbf{X}_1\mathbf{Y}_1^T+\lambda_2\mathbf{X}_2\mathbf{Y}_2^T+\cdots+\lambda_k\mathbf{X}_k\mathbf{Y}_k^T$$

$$=\lambda_1\mathbf{G}_1+\lambda_2\mathbf{G}_2+\cdots+\lambda_k\mathbf{G}_k.$$

For $\mathbf{G}_i=\mathbf{X}_i\mathbf{Y}_i^T$, the statement $\mathbf{P}\mathbf{P}^{-1}=\mathbf{I}$ translates to $\sum_{i=1}^k\mathbf{G}_i=\mathbf{I}$, and

$$\mathbf{P}^{-1}\mathbf{P}=\mathbf{I}\implies\mathbf{Y}_i^T\mathbf{X}_j=\begin{cases}\mathbf{I} & \text{when } i=j,\\ \mathbf{0} & \text{when } i\neq j,\end{cases}\implies\begin{cases}\mathbf{G}_i^2=\mathbf{G}_i,\\ \mathbf{G}_i\mathbf{G}_j=\mathbf{0} & \text{when } i\neq j.\end{cases}$$

Simple Eigenvalues and Projectors

If $\mathbf{x}$ and $\mathbf{y}^*$ are respective right-hand and left-hand eigenvectors associated with a *simple* eigenvalue $\lambda\in\sigma(\mathbf{A})$, then

$$\mathbf{G}=\mathbf{x}\mathbf{y}^*/\mathbf{y}^*\mathbf{x}$$

is the projector onto $N(\mathbf{A}-\lambda\mathbf{I})$ along $R(\mathbf{A}-\lambda\mathbf{I})$. In the context of the spectral theorem this means that $\mathbf{G}$ is the spectral projector associated with λ.

Summary of Diagonalizability

For an $n \times n$ matrix $\mathbf{A}$ with spectrum $\sigma(\mathbf{A}) = \{\lambda_1, \lambda_2, \ldots, \lambda_k\}$, the following statements are equivalent.

- $\mathbf{A}$ is similar to a diagonal matrix—i.e., $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{D}$.
- $\mathbf{A}$ has a complete linearly independent set of eigenvectors.
- Every λ_i is semisimple—i.e., $\mathit{geo\ mult}_{\mathbf{A}}(\lambda_i) = \mathit{alg\ mult}_{\mathbf{A}}(\lambda_i)$.
- $\mathbf{A} = \lambda_1\mathbf{G}_1 + \lambda_2\mathbf{G}_2 + \cdots + \lambda_k\mathbf{G}_k$, where
  - ▷ $\mathbf{G}_i$ is the projector onto $N(\mathbf{A} - \lambda_i\mathbf{I})$ along $R(\mathbf{A} - \lambda_i\mathbf{I})$,
  - ▷ $\mathbf{G}_i\mathbf{G}_j = \mathbf{0}$ whenever $i \neq j$,
  - ▷ $\mathbf{G}_1 + \mathbf{G}_2 + \cdots + \mathbf{G}_k = \mathbf{I}$,
  - ▷ $\mathbf{G}_i = \prod_{\substack{j=1 \\ j \neq i}}^{k} (\mathbf{A} - \lambda_j\mathbf{I}) \Big/ \prod_{\substack{j=1 \\ j \neq i}}^{k} (\lambda_i - \lambda_j)$
  - ▷ If λ_i is a simple eigenvalue associated with right-hand and left-hand eigenvectors $\mathbf{x}$ and $\mathbf{y}^*$, respectively, then $\mathbf{G}_i = \mathbf{x}\mathbf{y}^*/\mathbf{y}^*\mathbf{x}$.

Normal Matrices

- A matrix $\mathbf{A}$ is diagonalizable if and only if $\mathbf{A}$ possesses a complete independent set of eigenvectors, and is such a complete set is used for columns of $\mathbf{P}$, then $\mathbf{P}^{-1}\mathbf{A}\mathbf{P}$ is diagonal.
- But there's no guarantee that a complete orthonormal set of egenvector can be found.
- In other words, there's no assurance that $\mathbf{P}$ can be taken to be unitary (or orthogonal).
- When are complete orthonormal sets of eigenvectors produced?

Unitary Diagonalization

$\mathbf{A} \in \mathcal{C}^{n\times n}$ is unitarily similar to a diagonal matrix (i.e., $\mathbf{A}$ has a complete orthonormal set of eigenvectors) if and only if $\mathbf{A}^*\mathbf{A} = \mathbf{A}\mathbf{A}^*$, in which case $\mathbf{A}$ is said to be a ***normal matrix.***

- Whenever $\mathbf{U}^*\mathbf{A}\mathbf{U} = \mathbf{D}$ with $\mathbf{U}$ unitary and $\mathbf{D}$ diagonal, the columns of $\mathbf{U}$ must be a complete orthonormal set of eigenvectors for $\mathbf{A}$, and the diagonal entries of $\mathbf{D}$ are the associated eigenvalues.

- It's true that normal matrices possess a complete orthonormal set of eigenvectors.
- Not all complete independent sets of eigenvectors of a normal $\mathbf{A}$ are orthonormal.

Properties of Normal Matrices

If $\mathbf{A}$ is a normal matrix with $\sigma(\mathbf{A}) = \{\lambda_1, \lambda_2, \ldots, \lambda_k\}$, then

- $\mathbf{A}$ is RPN—i.e., $R(\mathbf{A}) \perp N(\mathbf{A})$
- Eigenvectors corresponding to distinct eigenvalues are orthogonal. In other words,

$$N(\mathbf{A} - \lambda_i \mathbf{I}) \perp N(\mathbf{A} - \lambda_j \mathbf{I}) \quad \text{for } \lambda_i \neq \lambda_j.$$

- The spectral theorems hold, but the spectral projectors specialize to become orthogonal projectors because $R(\mathbf{A} - \lambda_i \mathbf{I}) \perp N(\mathbf{A} - \lambda_i \mathbf{I})$.
- Several common types of matrices are normal. For example, real-symmetric and hermitian matrices, real skew-symmetric and skew-hermitian matrices, orthogonal and unitary matrices are normal.
- Real-symmetric and hermitian matrices have some special eigenvalue properties.

Symmetric and Hermitian Matrices

In addition to the properties inherent to all normal matrices,

- Real-symmetric and hermitian matrices have real eigenvalues.
- $\mathbf{A}$ is real symmetric if and only if $\mathbf{A}$ is *orthogonally* similar to a real-diagonal matrix $\mathbf{D}$—i.e., $\mathbf{P}^T\mathbf{A}\mathbf{P} = \mathbf{D}$ for some orthogonal $\mathbf{P}$.
- Real skew-symmetric and skew-hermitian matrices have pure imaginary eigenvalues.

Some results:

- Largest and Smallest Eigenvalues: Since the eigenvalues of a hermitian matrix $\mathbf{A}_{n\times n}$ are real, they can be ordered as $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$.
- Eigenvalue Perturbations. Suppose $\mathbf{A}$ is perturbed by a hermitian $\mathbf{E}$ with eigenvalues $\epsilon_1 \geq \epsilon_2 \geq \cdots \geq \epsilon_n$ to produce $\mathbf{B} = \mathbf{A} + \mathbf{E}$, which is also hermitian.
- Interlaced Eigenvalues. For $\mathbf{c} \in \mathcal{C}^{n\times 1}$, let $\mathbf{B}$ be the bordered matrix $\mathbf{B} = \begin{pmatrix} \mathbf{A} & \mathbf{c} \\ \mathbf{c}^* & \alpha \end{pmatrix}_{n+1\times n+1}$ with eigenvalues $\beta_1 \geq \beta_2 \geq \cdots \geq \beta_n \geq \beta_{n+1}$.The eigenvalues of $\mathbf{A}$ interlace with those of $\mathbf{B}$ in that

$$\beta_1 \geq \lambda_1 \geq \beta_2 \geq \lambda_2 \geq \cdots \geq \beta_n \geq \lambda_n \geq \beta_{n+1}$$

Singular Values and Eigenvalues

For $\mathbf{A} \in \mathcal{C}^{m\times n}$ with $rank(\mathbf{A}) = r$, the following statements are valid.

- The nonzero eigenvalues of $\mathbf{A}^*\mathbf{A}$ and $\mathbf{A}\mathbf{A}^*$ are equal and positive.
- The nonzero singular values of $\mathbf{A}$ are the positive square roots of the nonzero eigenvalues of $\mathbf{A}^*\mathbf{A}$ (and $\mathbf{A}\mathbf{A}^*$).
- If $\mathbf{A}$ is normal with nonzero eigenvalues $\{\lambda_1, \lambda_2, \ldots, \lambda_r\}$, then the nonzero singular values of $\mathbf{A}$ are $\{|\lambda_1|, |\lambda_2|, \ldots, |\lambda_r|\}$.
- Right-hand and left-hand singular vectors for $\mathbf{A}$ are special eigenvectors for $\mathbf{A}^*\mathbf{A}$ and $\mathbf{A}\mathbf{A}^*$, respectively.
- Any complete orthonormal set of eigenvectors $\mathbf{v}_i$ for $\mathbf{A}^*\mathbf{A}$ can serve as a complete set of right-hand singular vectors for $\mathbf{A}$, and a corresponding complete set of left-hand singular vectors is given by $\mathbf{u}_i = \mathbf{A}\mathbf{v}_i / \|\mathbf{A}\mathbf{v}_i\|_2$, $i = 1, 2, \ldots, r$, together with any orthonormal basis $\{\mathbf{u}_{r+1}, \mathbf{u}_{r+2}, \ldots, \mathbf{u}_m\}$ for $N(\mathbf{A}^*)$. Similarly, any complete orthonormal set of eigenvectors for $\mathbf{A}\mathbf{A}^*$ can be used as left-hand singular vectors for $\mathbf{A}$, and corresponding right-hand singular vectors can be built in an analogous way.
- The hermitian matrix $\mathbf{B} = \begin{pmatrix} \mathbf{0}_{m\times m} & \mathbf{A} \\ \mathbf{A}^* & \mathbf{0}_{n\times n} \end{pmatrix}$ of order $m + n$ has nonzero eigenvalues $\{\pm\sigma_1, \pm\sigma_2, \ldots, \pm\sigma_r\}$ in which $\{\sigma_1, \sigma_2, \ldots, \sigma_r\}$ are the nonzero singular values of $\mathbf{A}$.

Positive Definite Matrices

- Since the symmetric structure of a matrix forces its eigenvalues to be real, what additional property will force all eigenvalues to be positive?
- If $\mathbf{A} \in \Re^{n \times n}$ is symmetric, then, there is an orthogonal matrix $\mathbf{P}$ such that $\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^T$, where $\mathbf{D} = diag(\lambda_1, \lambda_2, \cdots, \lambda_n)$ is real.
- If $\lambda_i \geq 0$ for each i, then $\mathbf{D}^{1/2}$ exists, so

$$\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^T = \mathbf{P}\mathbf{D}^{1/2}\mathbf{D}^{1/2}\mathbf{P}^T = \mathbf{B}^T\mathbf{B} \quad for \quad \mathbf{B} = \mathbf{D}^{1/2}\mathbf{P}^T,$$

  and $\lambda_i > 0$ for each i if and only if $\mathbf{B}$ is nonsingular.
- Conversely, if $\mathbf{A}$ can be factored as $\mathbf{A} = \mathbf{B}^T\mathbf{B}$, then all eigenvalues of $\mathbf{A}$ are nonnegative because for any eigenpair $(\lambda, \mathbf{x})$,

$$\lambda = \frac{\mathbf{x}^T\mathbf{A}\mathbf{x}}{\mathbf{x}^T\mathbf{x}} = \frac{\mathbf{x}^T\mathbf{B}^T\mathbf{B}\mathbf{x}}{\mathbf{x}^T\mathbf{x}} = \frac{\|\mathbf{B}\mathbf{x}\|_2^2}{\|\mathbf{x}\|_2^2} \geq 0.$$

  Moreover, if $\mathbf{B}$ is nonsingular, then $N(\mathbf{B}) = 0 \Longrightarrow \mathbf{B}\mathbf{x} \neq 0$, so $\lambda > 0$.
- A real-symmetric matrix $\mathbf{A}$ has nonnegative eigenvalues if and only if $\mathbf{A}$ can be factored as $\mathbf{A} = \mathbf{B}^T\mathbf{B}$, and all eigenvalues are positive if and only if $\mathbf{B}$ is nonsingular.

- A symmetric matrix $\mathbf{A}$ whose eigenvalues are positive is called **positive definite**, and when the eigenvalues are just nonnegative, $\mathbf{A}$ is said to be **positive semidefinite**.

Positive Definite Matrices

For real-symmetric matrices $\mathbf{A}$, the following statements are equivalent, and any one can serve as the definition of a ***positive definite*** matrix.

- $\mathbf{x}^T\mathbf{A}\mathbf{x} > 0$ for every nonzero $\mathbf{x} \in \Re^{n\times 1}$ (most commonly used as the definition).
- All eigenvalues of $\mathbf{A}$ are positive.
- $\mathbf{A} = \mathbf{B}^T\mathbf{B}$ for some nonsingular $\mathbf{B}$.
  - ▷ While $\mathbf{B}$ is not unique, there is one and only one *upper-triangular* matrix $\mathbf{R}$ with positive diagonals such that $\mathbf{A} = \mathbf{R}^T\mathbf{R}$. This is the ***Cholesky factorization*** of $\mathbf{A}$.
- $\mathbf{A}$ has an LU (or LDU) factorization with all pivots being positive.
  - ▷ The LDU factorization is of the form $\mathbf{A} = \mathbf{L}\mathbf{D}\mathbf{L}^T = \mathbf{R}^T\mathbf{R}$, where $\mathbf{R} = \mathbf{D}^{1/2}\mathbf{L}^T$ is the ***Cholesky factor*** of $\mathbf{A}$.
- The leading principal minors of $\mathbf{A}$ are positive.
- All principal minors of $\mathbf{A}$ are positive.

For hermitian matrices, replace $(\star)^T$ by $(\star)^*$ and $\Re$ by $\mathcal{C}$.

- At first glance it's tempting to think that statements about positive definite matrices translate to positive semidefinite matrices simply by replacing the word positive by nonnegative, but this is not always true.
- When $\mathbf{A}$ has zero eigenvalues, there is no LU factorzation.

Positive Semidefinite Matrices

For real-symmetric matrices such that $rank(\mathbf{A}_{n\times n}) = r$, the following statements are equivalent, so any one of them can serve as the definition of a ***positive semidefinite*** matrix.

- $\mathbf{x}^T\mathbf{A}\mathbf{x} \geq 0$ for all $\mathbf{x} \in \Re^{n\times 1}$ (the most common definition).
- All eigenvalues of $\mathbf{A}$ are nonnegative.
- $\mathbf{A} = \mathbf{B}^T\mathbf{B}$ for some $\mathbf{B}$ with $rank(\mathbf{B}) = r$.
- All principal minors of $\mathbf{A}$ are nonnegative.

For hermitian matrices, replace $(\star)^T$ by $(\star)^*$ and $\Re$ by $\mathcal{C}$.

Quadratic Forms

For a vector $\mathbf{x} \in \Re^{n\times 1}$ and a matrix $\mathbf{A} \in \Re^{n\times n}$, the scalar function defined by

$$f(\mathbf{x}) = \mathbf{x}^T\mathbf{A}\mathbf{x} = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}x_ix_j$$

is called a ***quadratic form***. A quadratic form is said to be *positive definite* whenever $\mathbf{A}$ is a positive definite matrix.

- Because $\mathbf{x}^T\mathbf{A}\mathbf{x} = \mathbf{x}^T[\frac{(\mathbf{A}+\mathbf{A}^T)}{2}]\mathbf{x}$, and $\frac{(\mathbf{A}+\mathbf{A}^T)}{2}$ is symmetric, the matrix of a quadratic form can always be forced to be symmetric.
- For this reason, it is assumed that the matrix of every quadratic form is symmetric.
- When $\mathbf{x} \in \mathcal{C}^{n\times 1}$, $\mathbf{A} \in \mathcal{C}^{n\times n}$, and $\mathbf{A}$ is hermitian, the expression $\mathbf{x}^H\mathbf{A}\mathbf{x}$ is known as a complex quadratic form.
- **Diagonalization of a Quadratic Form.** A quadratic form $f(\mathbf{x}) = \mathbf{x}^T\mathbf{D}\mathbf{x}$ is said to be a diagonal form whenever $\mathbf{D}_{n\times n}$ is a diagonal matrix, in which case $\mathbf{x}^T\mathbf{D}\mathbf{x} = \sum_{i=1}^n d_{ii}x_i^2$.
  - Every quadratic form $\mathbf{x}^T\mathbf{A}\mathbf{x}$ can be diagonalized by making a change of variables (coordinates) $\mathbf{y} = \mathbf{Q}^T\mathbf{x}$, Where $\mathbf{Q}$ is an orthogonal matrix such that $\mathbf{Q}^T\mathbf{A}\mathbf{Q} = \mathbf{D} = diag(\lambda_1, \lambda_2, \cdots, \lambda_n)$, where $\lambda_i \in \sigma(\mathbf{A})$.
  - Setting $\mathbf{y} = \mathbf{Q}^T\mathbf{x}$ gives

$$\mathbf{x}^T\mathbf{A}\mathbf{x} = \mathbf{y}^T\mathbf{D}\mathbf{y} = \sum_{i=1}^n \lambda_i y_i^2.$$

  - The effect of diagonalizing a quadratic form in this way to rotate the standard coordinate system so that in the new coordinate system the graph of $\mathbf{x}^T\mathbf{A}\mathbf{x} = \alpha$ is in standard form.

- If $\mathbf{A}$ is positive definite, then all of its eigenvalues are positive, it clear that the graph of $\mathbf{x}^T\mathbf{A}\mathbf{x}$ for a constant $\alpha > 0$ is an ellipsoid centered at the origin.
- It's not necessary to solve an eigenvalue problem to diagonalize a quadratic form because a **congruence transformation** $\mathbf{C}^T\mathbf{A}\mathbf{C}$ in which $\mathbf{C}$ is nonsingular can be found that will do the job.
- A particularly convenient congruence transformation is produced by the LDU factorization for $\mathbf{A}$, which is $\mathbf{A} = \mathbf{L}\mathbf{D}\mathbf{L}^T$ because $\mathbf{A}$ is symmetric.
- This factorization is relatively cheap, and the diagonal entries in $\mathbf{D} = diag(p_1, p_2, \cdots, p_n)$ are the pivots that emerge during Gaussian elimination.
- Setting $\mathbf{y} = \mathbf{L}^T\mathbf{x}$ yields $\mathbf{x}^T\mathbf{A}\mathbf{x} = \mathbf{y}^T\mathbf{D}\mathbf{y} = \sum_{i=1}^n p_i y_i^2$.

Exercises

1. Determine the eigenvalues and eigenvectors for the following matrices.

$$\mathbf{A} = \begin{pmatrix} -10 & -7 \\ 14 & 11 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 3 & -2 & 5 \\ 0 & 1 & 4 \\ 0 & -1 & 5 \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} 0 & 6 & 3 \\ -1 & 5 & 1 \\ -1 & 2 & 4 \end{pmatrix}.$$

2. Explain why the eigenvalues of triangular and diagonal matrices

$$\mathbf{T} = \begin{pmatrix} t_{11} & t_{12} & \cdots & t_{1n} \\ 0 & t_{22} & \cdots & t_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & t_{nn} \end{pmatrix} \quad \textit{and} \quad \mathbf{D} = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}$$

   are simply the diagonal entries-the t_{ii}'s and λ_i's.

3. Prove that $0 \in \sigma(\mathbf{A})$ if and only if $\mathbf{A}$ is a singular matrix.

4. For $\mathbf{T} = \begin{pmatrix} \mathbf{A} & mb \\ 0 & \mathbf{C} \end{pmatrix}$, prove $\det(\mathbf{T} - \lambda\mathbf{I}) = \det(\mathbf{A} - \lambda\mathbf{I})\det(\mathbf{C} - \lambda\mathbf{I})$ to conclude that $\sigma(\mathbf{T}) = \sigma(\mathbf{A}) \bigcup \sigma(\mathbf{C})$ for square $\mathbf{A}$ and $\mathbf{C}$.

5. If $\mathbf{A}$ is nonsingular, and if $(\lambda, \mathbf{x})$ is an eigenpair for $\mathbf{A}$, show that $(\lambda^{-1}, \mathbf{x})$ is an eigenpair for $\mathbf{A}^{-1}$.
6. Diagonalize $\mathbf{A} = \begin{pmatrix} -8 & -6 \\ 12 & 10 \end{pmatrix}$ with a similarity transformation, or else explain why $\mathbf{A}$ can't be diagonalized.
7. (a)Verify that $alg\ mult_{\mathbf{A}}(\lambda) = geo\ mult_{\mathbf{A}}(\lambda)$ for each eigenvalue of

$$\mathbf{A} = \begin{pmatrix} -4 & -3 & -3 \\ 0 & -1 & 0 \\ 6 & 6 & 5 \end{pmatrix}.$$

(b) Find a nonsingular $\mathbf{P}$ such that $\mathbf{P}^{-1}\mathbf{A}\mathbf{P}$ is a diagonal matrix.
8. Is $\mathbf{A} = \begin{pmatrix} 5+i & -2i \\ 2 & 4+2i \end{pmatrix}$ a normal matrix?
9. Explain why a triangular matrix is normal if and only if it ie diagonal.
10. which of the following matrices are positive definite?

$$\mathbf{A} = \begin{pmatrix} 1 & -1 & -1 \\ -1 & 5 & 1 \\ -1 & 1 & 5 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 20 & 6 & 8 \\ 6 & 3 & 0 \\ 8 & 0 & 8 \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} 2 & 0 & 2 \\ 0 & 6 & 2 \\ 2 & 2 & 4 \end{pmatrix}.$$